Definition (Polarization):

Let (M, ω) be a symplectic manifold of dimension $2n$ and $TM_{\mathbb{C}}$ the complexified tangent bundle.

$V_P \subset TM_{\mathbb{C}}$ subbundle is "integrable"

if: for $X, Y : M \to V_P \Rightarrow [X, Y] : M \to V_P$

V_P is "Lagrangian" if

$$\forall x \in M : \dim (V_P)_x = n \quad \text{and}$$
$$\omega|_{(V_P)_x} = 0$$

A Lagrangian V_P is called "polarization", if it is integrable.

Define $\mathcal{H}_P = \{ s \in \mathcal{H} \mid \nabla_X s = 0, X \in \Gamma(M, V_P) \}$

$\uparrow$ quantum Hilbert space

Definition (Kähler polarization):

Let (M, ω) be a Kähler manifold, set $V_P = TM^{(0,1)}$

$\longrightarrow \mathcal{H}_P = H^0(M, L)$ space of holomorphic sections

§1 Loop groups and affine Lie algebras

Definition (Loop group)

Let G be a compact connected Lie group.
Define $LG \equiv \{ \gamma : S^1 \to G \mid \gamma \text{ smooth map} \}$
where $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$

group structure on LG:

$$(\gamma_1 \cdot \gamma_2)(z) = \gamma_1(z)\gamma_2(z), \quad \gamma_1, \gamma_2 \in LG$$

$\to$ $(\gamma_1, \gamma_2) \mapsto \gamma_1 \cdot \gamma_2$ and $\gamma \mapsto \gamma^{-1}$ are smooth maps

$\Rightarrow$ LG is infinite dimensional Lie group "Loop group" of G

In these lectures: $G = SU(2)$

Let $\mathbb{C}((t))$ denote the $\mathbb{C}$ algebra of the Laurent series: $f(t) = \sum_{n=-m}^{\infty} a_n t^n$, $m \in \mathbb{Z}$

Let g be the complexified Lie algebra of G
$\to$ in our case: $g = sl_2(\mathbb{C})$. Set

$$Lg = g \otimes \mathbb{C}((t)) \quad (\text{compl. Lie algebra of } LG)$$

Lie bracket:

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg$$

Now define $\hat{\mathfrak{g}}$ as the direct sum

$$L\mathfrak{g} \oplus \mathbb{C}c$$

$\uparrow$ one-dim complex vector space with basis c

Lie-bracket for $\hat{\mathfrak{g}}$:

$$[\xi + \alpha c, \eta + \beta c] = [\xi, \eta] + \omega(\xi, \eta)\, c, \qquad (*)$$

$$\xi, \eta \in L\mathfrak{g} \quad \alpha, \beta \in \mathbb{C}$$

where $\omega: L\mathfrak{g} \times L\mathfrak{g} \to \mathbb{C}$ bilinear form

$\to$ c belongs to center of $\hat{\mathfrak{g}}$:

$$[\xi, c] = [\xi, 0 + c] = [\xi, 0] + \underbrace{\omega(\xi, 0)}_{=0} c = 0$$

$\to$ $(*)$ defines a Lie algebra structure on $\hat{\mathfrak{g}}$ <u>iff</u> for $x, y, z \in L\mathfrak{g}$:

(a) $\omega(x, y) = -\omega(y, x)$ (anti-symmetry)

(b) $\omega([x,y], z) + \omega([y,z], x) + \omega([z,x], y) = 0$ (Jacobi identity)

$\to$ Lie bracket for $\hat{\mathfrak{g}}$:

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + \omega(X \otimes f, Y \otimes g)\, c$$

Condition (b) is called "2-cocycle condition"

ω and ω' are equiv. iff $\exists\ \mu: L\mathfrak{g} \to \mathbb{C}$ linear s.t. $\omega(x,y) = \omega'(x,y) + \mu([x,y])$ $\forall\ x, y \in L\mathfrak{g}$

$\to$ $\hat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}c$ is Lie algebra with $c \in \mathrm{Center}(\hat{\mathfrak{g}})$

Definition:

The Lie algebra $\hat{g}$ is called "central extension" of Lg.

Definition (Lie algebra cohomology):

For a Lie algebra a and left a module M define

$$C^p(a, M) = \mathrm{Hom}_{\mathbb{C}}\left(\bigwedge^p a, M\right)$$

(p-th cochain group)

and differential $d_p : C^p(a, M) \to C^{p+1}(a, M)$:

$$(d\omega)(x_0, x_1, \dots, x_p)$$

$$= \sum_{i=0}^{p} (-1)^i x_i \, \omega(x_0, \dots, \hat{x}_i, \dots, x_p)$$

$$+ \sum_{0 \le i < j \le p} (-1)^{i+j} \omega([x_i, x_j], x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_p)$$

for $\omega \in C^p(a, M)$. Then

$$H^p(a, M) = \mathrm{Ker}\, d_p / \mathrm{Im}\, d_{p-1}$$

is called p-th cohomology of a with coefficients in M.

Regarding $\mathbb{C}$ as a trivial g module $(g\,\mathbb{C} = 0)$

$\longrightarrow \omega \in H^2(Lg, \mathbb{C})$ (condition (b) becomes $d\omega = 0$ and $\mu([x, y])$ $= d\sigma$ for $\sigma \in H^1(Lg, \mathbb{C}$)

The converse can also by shown

$$\rightarrow \left\{ \begin{matrix} \text{isomorphism classes} \\ \text{of central extensions of } Lg \end{matrix} \right\} \longleftrightarrow H^2(Lg, \mathbb{C})$$

<u>Definition</u> (Cartan-Killing form):

A non-degenerate symmetric bilinear form

$$\langle \, , \, \rangle : g \times g \to \mathbb{C}$$

satisfying

$$\langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle \qquad (**)$$

for $X, Y, Z \in g$ is called "Cartan-Killing form"

For $g = sl_2(\mathbb{C})$ we set $\langle X, Y \rangle = \mathrm{Tr}(XY)$

<u>Proposition 1</u>:

For the loop algebra Lg we have $H^2(Lg, \mathbb{C}) \cong \mathbb{C}$. The generator of $H^2(Lg, \mathbb{C})$ is given by

$$\omega(X \otimes f, Y \otimes g) = \langle X, Y \rangle \, \mathrm{Res}_{t=0}(df\, g)$$

where $\mathrm{Res}_{t=0}\left(\sum_n c_n t^n dt\right) = c_{-1}$

<u>Proof:</u>

$G \subset LG$ by choosing $\gamma : S^1 \to G$ constant

For $g \in G$ write $g = \exp t Z$, $Z \in g$. For $X \in Lg$ we have $gXg^{-1} = X + t[Z, X] + O(t^2)$

$\rightarrow$ For α 2-cocycle of Lg we have

$$\lim_{t\to 0}\frac{1}{t}\left[\alpha(gXg^{-1}, gYg^{-1}) - \alpha(X,Y)\right] = \alpha(Z,[X,Y])$$

for $X, Y \in Lg$ by 2-cocycle condition.

Define 1-cochain $\mu_Z : Lg : \to \mathbb{C}$ by

$$\mu_Z(u) = \alpha(Z, u) \quad \text{for } Z \in Lg$$

$$\Rightarrow \alpha(Z,[X,Y]) = \mu_Z([X,Y]) = d\mu_Z(X,Y)$$

trivial in $H^2(Lg, \mathbb{C})$

Denote $\alpha_g(X,Y) = \alpha(gXg^{-1}, gYg^{-1})$. Then

$$\int_G \alpha_g \, dg$$

in invariant under conjugation and is cohomologous to α (G is simply connected).

$\rightarrow$ suppose that α is invariant under conj.:

$$\alpha([Z,X],Y) + \underbrace{\alpha(X,[Z,Y])}_{= -\alpha(X,[Y,Z])} = 0 \rightarrow (**)$$

Set $\alpha_{m,n}(X,Y) = \alpha(X\otimes t^m, Y\otimes t^n)$ for $X, Y \in g$

$\rightarrow \alpha_{m,n} : g\times g \to \mathbb{C}$ is bilinear and satisfies $(**)$

$\rightarrow \alpha_{m,n}$ is symmetric (and therefore Killing-form) since g is simple.

Then $\alpha_{m,n} = -\alpha_{n,m}$ (α anti-sym.)

Cocycle condition for α becomes

$$\alpha_{m+n,p} + \alpha_{n+p,m} + \alpha_{p+m,n} = 0$$

$n = p = 0 \rightarrow \alpha_{m,0} = 0 \ \forall m$

$p = -m-n \longrightarrow \alpha_{m+n,-m-n} = \alpha_{m,-m} + \alpha_{n,-n}$

$\Rightarrow \alpha_{m,-m} = m\,\alpha_{1,-1}$

$p = q-m-n \longrightarrow \alpha_{q-m-n,m+n} = \alpha_{q-m,m} + \alpha_{q-n,n}$

$\Rightarrow \alpha_{q-k,k} = k\,\alpha_{q-1,1}$

$\Longrightarrow \alpha_{m,n} = 0$ if $m+n \neq 0$ $\left(q\alpha_{q-1,1} = \alpha_{0,q} = 0 \right)$

$\Longrightarrow \alpha_{m,n} = m\,\delta_{m+n,0}\,\alpha_{1,-1}$

$\alpha_{1,-1} : g \times g \rightarrow \mathbb{C}$ is g invariant sym. bilinear form $\rightarrow$ equal to Cartan-Killing form up to const.

set $\omega = \alpha_{1,-1}$

It can be easily shown that ω is not coboundary (exercise). $\square$

<u>Definition</u> (affine Lie algebra):

The central extension $\hat{g}$ of g with Lie bracket

$$[X \otimes t^m, Y \otimes t^n] = [X,Y] \otimes t^{m+n} + \langle X,Y \rangle m\, \delta_{m+n,0}\, c$$

for $X, Y \in g$ is called "affine Lie algebra" associated with g.